Statistical inference for Weibull tail-distributions

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Outline

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1. Weibull tail-distributions.

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables with cumulative distribution function F such that

(A1) $1 - F(x) = \exp(-H(x))$, $H^{-1}(t) = \inf\{x,\ H(x) \geq t\} = t^\theta \ell(t)$,

where

- $\theta > 0$ is the Weibull tail-coefficient,
- ℓ is a slowly varying function *i.e.*

$$\ell(\lambda x)/\ell(x) \to 1 \text{ as } x \to \infty \text{ for all } \lambda > 0.$$

The inverse failure rate function H^{-1} is said to be regularly varying at infinity with index θ and this property is denoted by $H^{-1} \in \mathcal{R}_\theta$.

In the following, we also assume a second order condition on ℓ:

(A2) There exist $\rho \leq 0$ and $b(x) \to 0$ such that uniformly locally on $\lambda \geq 1$

$$\log\left(\frac{\ell(\lambda x)}{\ell(x)}\right) \sim b(x) K_\rho(\lambda), \text{ when } x \to \infty,$$

with

$$K_\rho(\lambda) = \int_1^\lambda u^{\rho-1} du.$$

It can be shown that necessarily $|b| \in \mathcal{R}_\rho$. The second order parameter $\rho \leq 0$ tunes the rate of convergence of $\ell(\lambda x)/\ell(x)$ to 1. The closer ρ is to 0, the slower is the convergence.

Examples:

	θ	$\ell(x)$	$b(x)$	ρ
Absolute Gaussian $\lvert\mathcal{N}\rvert(\mu,\sigma^2)$	$1/2$	$2^{1/2}\sigma - \frac{\sigma}{2^{3/2}}\frac{\log x}{x} + O(1/x)$	$\frac{1}{4}\frac{\log x}{x}$	-1
Gamma $\Gamma(\alpha \neq 1, \beta)$	1	$\frac{1}{\beta} + \frac{\alpha-1}{\beta}\frac{\log x}{x} + O(1/x)$	$(1-\alpha)\frac{\log x}{x}$	-1
Weibull $\mathcal{W}(\alpha,\lambda)$	$1/\alpha$	λ	0	$-\infty$

Related work: Berred (1991), Broniatowski (1993), Beirlant, Broniatowski, Teugels, Vynckier, (1995), Beirlant, Bouquiaux, Werker (2006).

2. Kernel estimators of the Weibull tail-coefficient.

Principle: Our approach is based on the following approximation. Denoting by $q(t)$ the quantile function

$$q(t) = F^{-1}(1-t) = H^{-1}(\log(1/t)) = (\log(1/t))^{\theta}\ell(\log(1/t)),$$

we obtain for t and s small:

$$\begin{aligned} \log(q(t)) - \log(q(s)) &= \theta(\log_2(1/t) - \log_2(1/s)) + \log\left(\frac{\ell(\log(1/t))}{\ell(\log(1/s))}\right) \\ &\simeq \theta(\log_2(1/t) - \log_2(1/s)), \end{aligned} \tag{1}$$

where $\log_2(x) = \log\log(x)$ and since $\ell \in \mathcal{R}_0$. Considering $t = i/n$, $s = k_n/n$ and replacing F by its empirical counterpart yield

$$\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n}) \simeq \theta(\log_2(n/i) - \log_2(n/k_n)),$$

for $i = 1, \ldots, k_n - 1$ and where k_n is an intermediate sequence.

Our method: Estimation via linear combination of upper order statistics.

$$\hat{\theta}_n(\alpha) = \sum_{i=1}^{k_n-1} \alpha_{i,n}(\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n})) \Bigg/ \sum_{i=1}^{k_n-1} \alpha_{i,n}(\log_2(n/i) - \log_2(n/k_n)),$$

where

- $\alpha_{i,n} = W(i/k_n) + \varepsilon_{i,n}$,
- $\varepsilon_{i,n}$, $i = 1, \ldots, k_n - 1$ is a non-random sequence, and
- $W : [0,1] \to \mathbb{R}$ is a smooth score function, verifying

(A3) W has a continuous derivative W' on $(0,1)$,

(A4) There exist $M > 0$, $0 \leq q < 1/2$ and $p < 1$ such that, for all $x \in (0,1)$:
$|W(x)| \leq Mx^{-q}$ and $|W'(x)| \leq Mx^{-p-q}$.

Asymptotic normality: Suppose **(A1)**–**(A4)** hold. Then

$$k_n^{1/2}(\hat{\theta}_n(\alpha) - \theta) \xrightarrow{d} \mathcal{N}(0, \sigma^2(\theta, W)),$$

for any sequence (k_n) satisfying $k_n \to \infty$ and

$$k_n^{1/2} \max\{b(\log(n/k_n)), 1/\log(n/k_n), \|\varepsilon\|_{n,\infty}\} \to 0,$$

where we have defined:

$$\begin{aligned}
\|\varepsilon\|_{n,\infty} &= \max_{i=1,\dots,k_n-1} |\varepsilon_{i,n}|, \\
\mu(W) &= \int_0^1 W(x)\log(1/x)dx, \\
\sigma^2(W) &= \int_0^1 \int_0^1 W(x)W(y)\frac{\min(x,y)(1-\max(x,y))}{xy}dxdy, \\
\sigma(\theta, W) &= \theta\frac{\sigma(W)}{\mu(W)}.
\end{aligned}$$

Example 1. Constant weights $\alpha_{i,n} = 1$ for all $i = 1, \ldots, k_n - 1$ yield an existing estimator (Girard, 2004):

$$\hat{\theta}_n^G = \sum_{i=1}^{k_n-1} (\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n})) \Bigg/ \sum_{i=1}^{k_n-1} (\log_2(n/i) - \log_2(n/k_n)) \, .$$

We found back the same limiting result: If **(A1)** and **(A2)** hold then

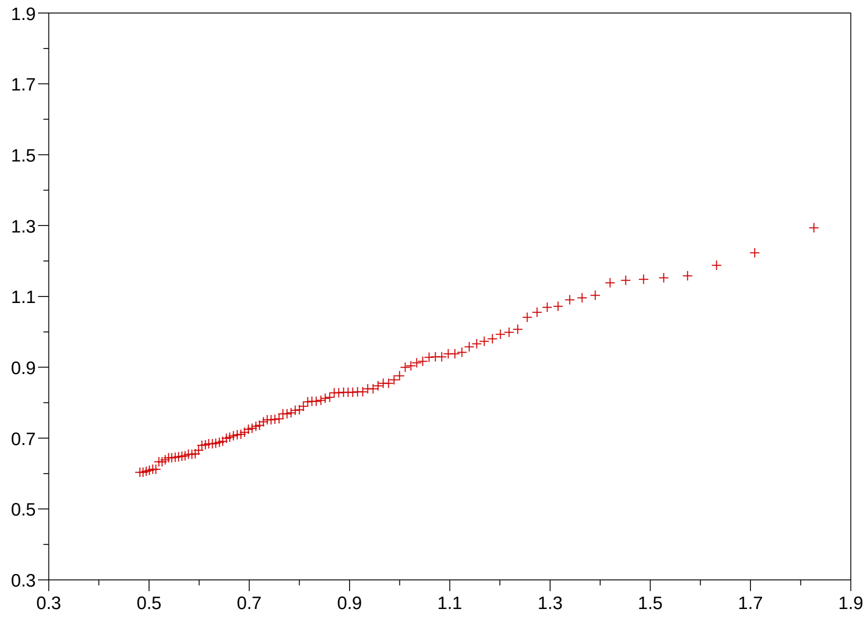
$$k_n^{1/2}(\hat{\theta}_n^G - \theta) \xrightarrow{d} \mathcal{N}(0, \theta^2),$$

for any sequence (k_n) satisfying $k_n \to \infty$ and

$$k_n^{1/2} \max\{b(\log(n/k_n)), 1/\log(n/k_n)\} \to 0.$$

Example 2: A new estimator of the Weibull tail-coefficient based on a QQ-plot.

- Drawing the pairs $(\log_2(n/i), \log(X_{n-i+1,n}))$ for $i = 1, \dots, k_n$ gives a graph which is approximatively linear (with slope θ).
- Example : $|\mathcal{N}|(0,1)$ distribution, $n = 500$, $k_n = 100$.



- $\hat{\theta}_n^Z$ is the least square estimator of θ based on the k_n largest observations:

$$\hat{\theta}_n^Z = \sum_{i=1}^{k_n-1} (\log_2(n/i) - \tau_n) \log(X_{n-i+1,n}) \Bigg/ \sum_{i=1}^{k_n-1} (\log_2(n/i) - \tau_n) \log_2(n/i) \,,$$

where

$$\tau_n = \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} \log_2(n/i) \,.$$

- Similar to the Zipf estimator for the extreme value index proposed by Kratz, Resnick (1996) and Schultze, Steinebach (1996).
- Particular case of $\hat{\theta}_n(\alpha)$ with $W(x) = -(\log(x) + 1)$. Thus, under **(A1)** and **(A2)**,

$$k_n^{1/2}(\hat{\theta}_n^Z - \theta) \xrightarrow{d} \mathcal{N}(0, 2\theta^2),$$

for any sequence (k_n) satisfying $k_n \to \infty$ and

$$k_n^{1/2} \max\{b(\log(n/k_n)), \log^2(k_n)/\log(n/k_n)\} \to 0.$$

3. Bias-reduced estimator of the Weibull tail-coefficient.

Principle: We focus on the case where the convergence in **(A2)** is slow, *i.e.*

(A5) $x|b(x)| \to \infty$ as $x \to \infty$.

Let us note that this condition implies $\rho \geq -1$. Gamma and (absolute) Gaussian distribution fulfill **(A5)** whereas Weibull distribution do not. Condition **(A2)** can be used to precise approximation (1):

$$\log(q(t)) - \log(q(s)) = \theta(\log_2(1/t) - \log_2(1/s)) + b(\log(1/s))K_\rho\left(\frac{\log(1/t)}{\log(1/s)}\right)(1 + o(1)). \quad (2)$$

Exponential regression models:

- Define $Z_i = i \log(n/i)(\log X_{n-i+1,n} - \log X_{n-i,n})$, $i = 1, \dots, k_n$. Then, under **(A1)**, **(A2)**, **(A5)**,

$$\sup_{1 \le i \le k_n} \left| Z_i - \left(\theta + b(\log(n/k_n)) \left(\frac{\log(n/i)}{\log(n/k_n)} \right)^{\rho} \right) f_i \right| = o_{\mathbb{P}}\left(b(\log(n/k_n)) \right), \tag{3}$$

for any sequence (k_n) such that $k_n \to \infty$ and $\log k_n / \log n \to 0$, and where $(f_1, \ldots, f_{k_n})$ is a vector of independent and standard exponentially distributed random variables.

- Similar to the ones proposed by Beirlant, Dierckx, Goegebeur, Matthys (1999), Feuerverger, Hall (1999) and Beirlant, Dierckx, Guillou, Starica (2002) in the case of Pareto-type distributions.

- One can plug the canonical choice $\rho = -1$ in the regression model (3) without perturbing the approximation so that

$$\sup_{1 \le i \le k_n} \left| Z_i - \left(\theta + b(\log(n/k_n)) \frac{\log(n/k_n)}{\log(n/i)} \right) f_i \right| = o_{\mathbb{P}}\left(b(\log(n/k_n)) \right). \tag{4}$$

Application 1: Bias-reduced estimator. Estimation of θ and $b(\log(n/k_n))$ by a Least-Square method after substituting ρ with the value -1:

$$\hat{\theta}_n^R = \overline{Z}_{k_n} - \widehat{b}(\log(n/k_n))\overline{x}_{k_n}, \quad \widehat{b}(\log(n/k_n)) = \frac{\sum_{i=1}^{k_n}(x_i - \overline{x}_{k_n})Z_i}{\sum_{i=1}^{k_n}(x_i - \overline{x}_{k_n})^2}, \tag{5}$$

where $x_i = \log(n/k_n)/\log(n/i)$, $\overline{x}_{k_n} = \frac{1}{k_n}\sum_{i=1}^{k_n} x_i$ and $\overline{Z}_{k_n} = \frac{1}{k_n}\sum_{i=1}^{k_n} Z_i$.

Asymptotic normality under **(A1)**, **(A2)**, **(A5)**:

$$\frac{k_n^{1/2}}{\log(n/k_n)}(\hat{\theta}_n^R - \theta) \xrightarrow{d} \mathcal{N}(0, \theta^2)$$

for any sequence (k_n) such that $k_n \to \infty$ and

$$k_n^{1/2} b(\log(n/k_n))/\log(n/k_n) \to \Lambda \neq 0.$$

Application 2: Adaptive selection of k_n. Adapted from Matthys, Beirlant (2003) in the context of extreme-value index estimation. Neglecting the bias correction in (5) yields the Maximum Likelihood estimator:

$$\hat{\theta}_n^{ML} = \overline{Z}_{k_n} = \frac{1}{k_n}\sum_{i=1}^{k_n} i\log(n/i)(\log X_{n-i+1,n} - \log X_{n-i,n}).$$

From (4), the Asymptotic Mean Squared Error (AMSE) associated to $\hat{\theta}_n^{ML}$ is given by

$$AMSE(\hat{\theta}_n^{ML}) = \frac{\theta^2}{k_n} + \left(b(\log(n/k_n))\frac{1}{k_n}\sum_{i=1}^{k_n}\frac{\log(n/k_n)}{\log(n/i)}\right)^2,$$

and can be estimated by

$$\widehat{AMSE}(\hat{\theta}_n^{ML}) = \frac{(\hat{\theta}_n^{R})^2}{k_n} + \left(\hat{b}(\log(n/k_n))\frac{1}{k_n}\sum_{i=1}^{k_n}\frac{\log(n/k_n)}{\log(n/i)}\right)^2.$$

The minimization of $\widehat{AMSE}(\hat{\theta}_n^{ML})$ with respect to k_n gives rise to an adaptive selection procedure.

4. Estimation of extreme quantiles.

Principle: An extreme quantile x_{p_n} of order $p_n < 1/n$ is defined by:

$$1 - F(x_{p_n}) = p_n.$$

Recall that, for small t and s

$$\frac{q(t)}{q(s)} = \frac{H^{-1}(\log(1/t))}{H^{-1}(\log(1/s))} = \left(\frac{\log(1/t)}{\log(1/s)}\right)^{\theta} \frac{\ell(\log(1/t))}{\ell(\log(1/s))} \simeq \left(\frac{\log(1/t)}{\log(1/s)}\right)^{\theta}.$$

Considering $t = p_n$, $s = k_n/n$, replacing F by its empirical counterpart and θ by an estimator $\hat{\theta}_n$ yield the following estimator

$$\hat{x}_{p_n}(\hat{\theta}_n) = X_{n-k_n+1,n} \left(\frac{\log(1/p_n)}{\log(n/k_n)}\right)^{\hat{\theta}_n} = X_{n-k_n+1,n} \exp\left(\hat{\theta}_n \log \tau_n\right),$$

where we have defined $\tau_n = \log(1/p_n)/\log(n/k_n)$.

Asymptotic normality: Based on the following result (Gardes, Girard, 2005). Suppose **(A1)** and **(A2)** hold. If morever

$$1 \leq \liminf \tau_n \leq \limsup \tau_n < \infty,$$

and

$$k_n^{1/2}(\hat{\theta}_n - \theta) \xrightarrow{d} \mathcal{N}(0, \sigma^2),$$

then

$$\frac{\log(n/k_n) k_n^{1/2}}{\log(k_n/(np_n))} \left(\frac{\hat{x}_{p_n}(\hat{\theta}_n)}{x_{p_n}} - 1 \right) \xrightarrow{d} \mathcal{N}(0, \sigma^2),$$

for any sequence (k_n) satisfying $k_n \to \infty$, $k_n/n \to 0$ and

$$k_n^{1/2} b(\log(n/k_n)) \to 0.$$

As a consequence, we obtain the asymptotic normality of $\hat{x}_{p_n}(\hat{\theta}_n^G)$, $\hat{x}_{p_n}(\hat{\theta}_n^Z)$ and $\hat{x}_{p_n}(\hat{\theta}_n^{\text{BBTV}})$ where

$$\hat{\theta}_n^{\text{BBTV}} = \frac{\log(n/k_n)}{X_{n-k_n+1,n}} \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} (X_{n-i+1,n} - X_{n-k_n+1,n})$$

is the estimator introduced by Beirlant, Broniatowski, Teugels, Vinckier (1995).

Bias-reduced estimator: Basing on the refined approximation (2) of $\log q(t) - \log q(s)$, it is natural to introduce

$$\hat{x}_{p_n}^R = X_{n-k_n+1,n} \exp\left(\hat{\theta}_n^R \log \tau_n + \hat{b}(\log(n/k_n)) K_{-1}(\tau_n)\right).$$

Asymptotic normality under **(A1)**, **(A2)**, **(A5)**:

$$\frac{k_n^{1/2}}{\log(n/k_n)} \left(\frac{\hat{x}_{p_n}^R}{x_{p_n}} - 1\right) \xrightarrow{d} \mathcal{N}(\Lambda\mu(\tau), \theta^2\sigma^2(\tau))$$

for any sequence (k_n) such that $k_n \to \infty$, $\tau_n \to \tau > 1$ and

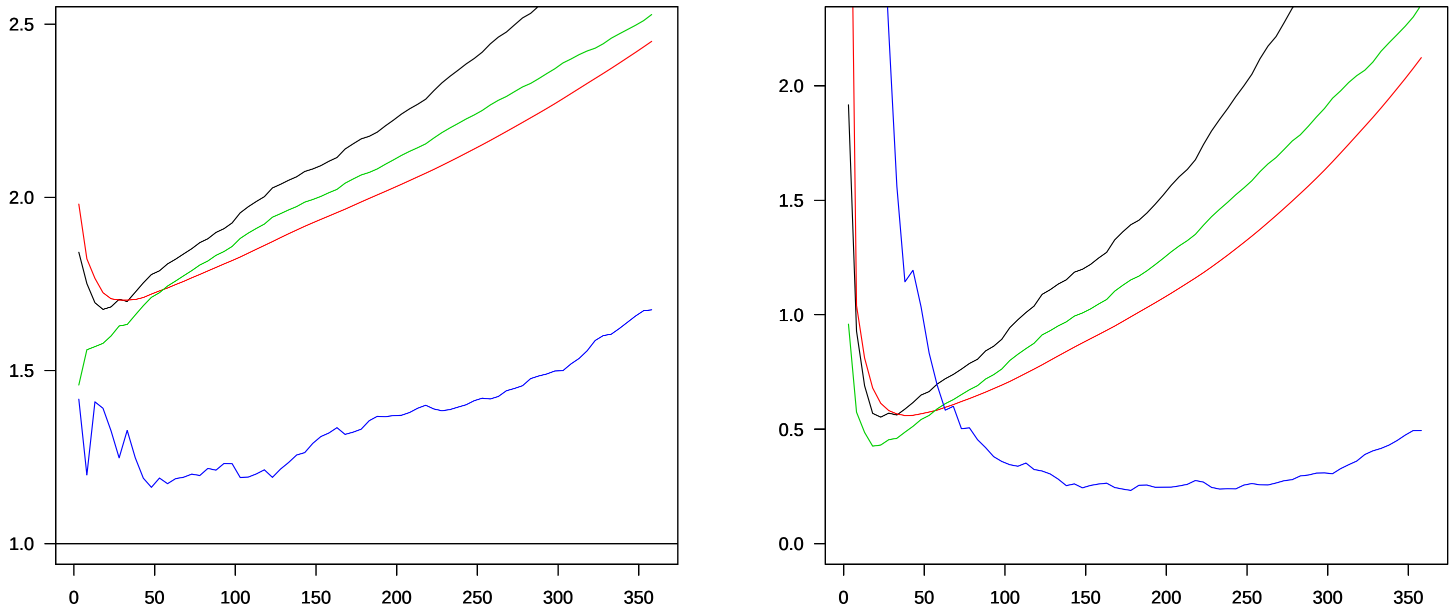
$$k_n^{1/2} b(\log(n/k_n)) / \log(n/k_n) \to \Lambda \neq 0,$$

where $\mu(\tau) = (K_{-1}(\tau) - K_\rho(\tau))$ and $\sigma^2(\tau) = (K_{-1}(\tau) - \log(\tau))^2$.

5. Simulation study.

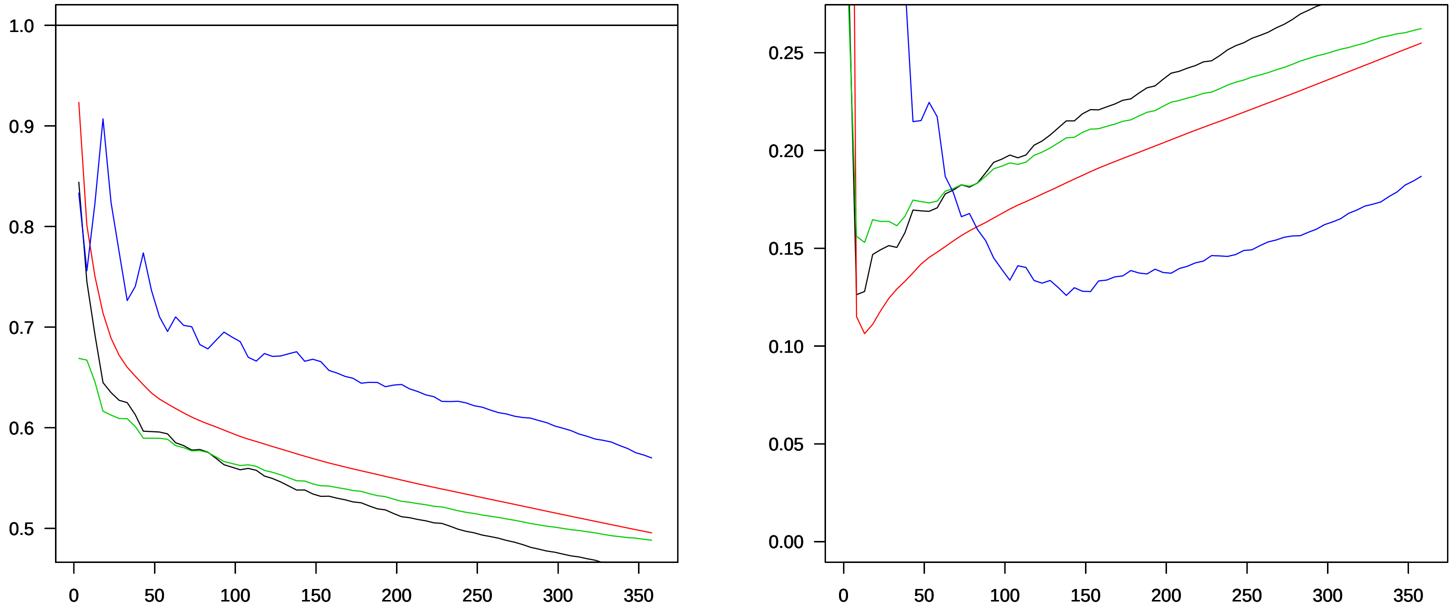
- Comparison of $\hat{\theta}_n^G$ (in black), $\hat{\theta}_n^Z$ (in red), $\hat{\theta}_n^{ML}$ (in green) and $\hat{\theta}_n^R$ (in blue) to the true θ (black horizontal line).
- Simulated distributions : Absolute Gaussian $|\mathcal{N}|(0,1)$, Gamma $\Gamma(0.25,1)$, $\Gamma(4,1)$, and Weibull $\mathcal{W}(4,4)$, $\mathcal{W}(0.25,0.25)$.
- Sample size $n = 500$, $k_n \in \{2,\dots,360\}$, 100 replications.
- Computation of the mean estimate (Hill plot, left pannel) and of the Mean Square Error (MSE, right pannel).

Gamma $\Gamma(0.25, 1) \longrightarrow \theta = 1,\ b(x) > 0$



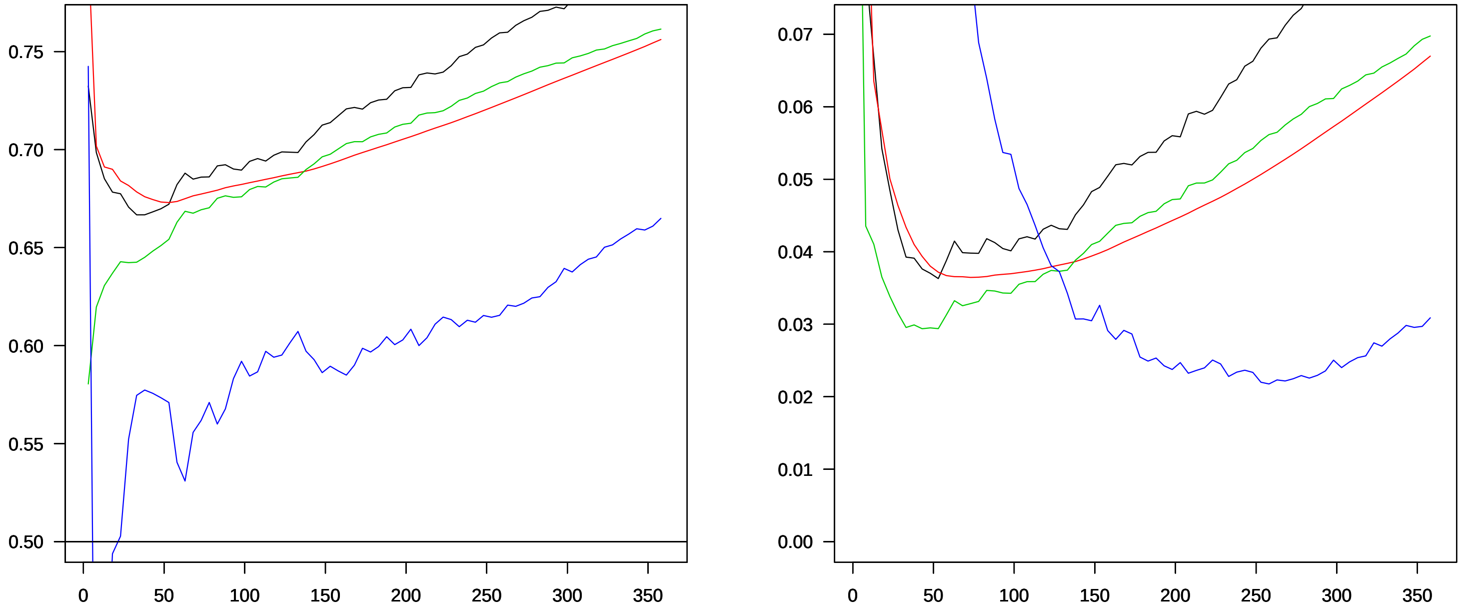
$\hat{\theta}_n^G$ (in black), $\hat{\theta}_n^Z$ (in red), $\hat{\theta}_n^{ML}$ (in green) and $\hat{\theta}_n^R$ (in blue)

Gamma $\Gamma(4,1) \longrightarrow \theta = 1,\ b(x) < 0$



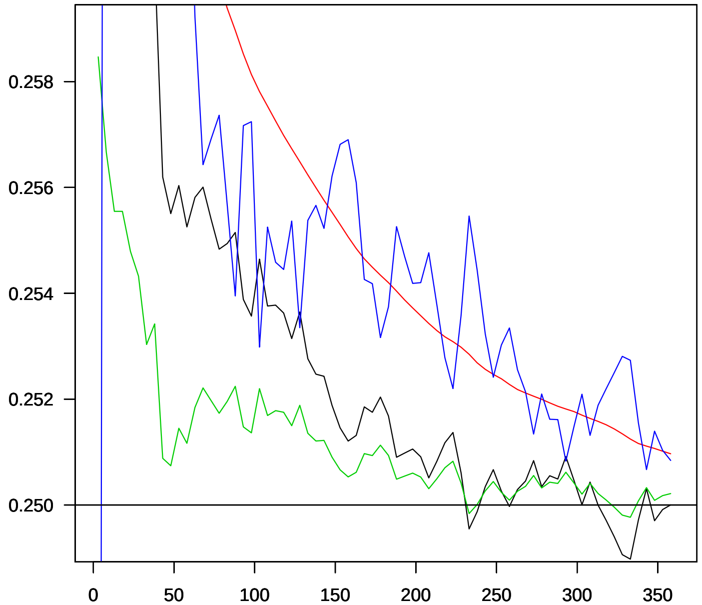
$\hat{\theta}_n^G$ (in black), $\hat{\theta}_n^Z$ (in red), $\hat{\theta}_n^{ML}$ (in green) and $\hat{\theta}_n^R$ (in blue)

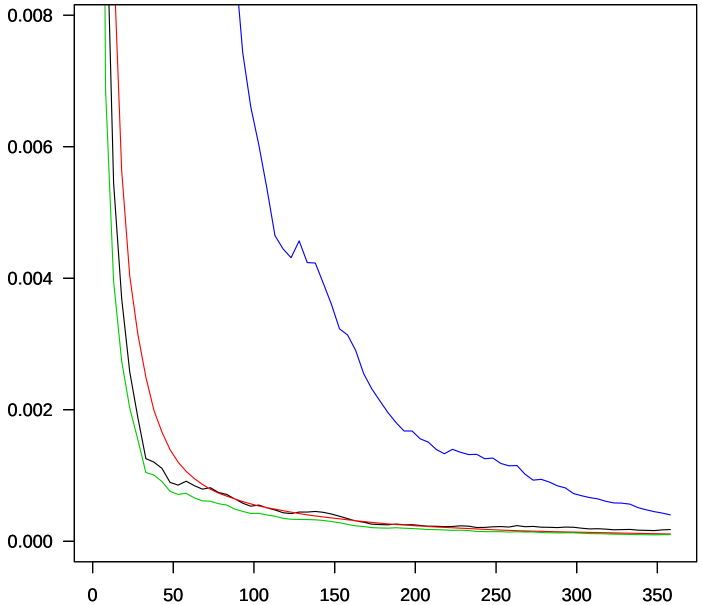
Absolute Gaussian $|\mathcal{N}|(0,1) \longrightarrow \theta = 1/2,\ b(x) > 0$



$\hat{\theta}_n^G$ (in black), $\hat{\theta}_n^Z$ (in red), $\hat{\theta}_n^{ML}$ (in green) and $\hat{\theta}_n^R$ (in blue)

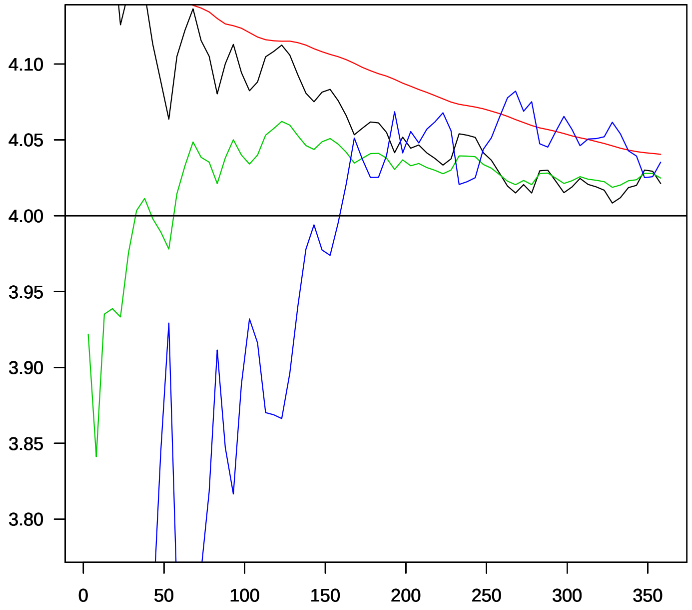
$$\text{Weibull } \mathcal{W}(4,4) \longrightarrow \theta = 0.25,\ b(x) = 0$$

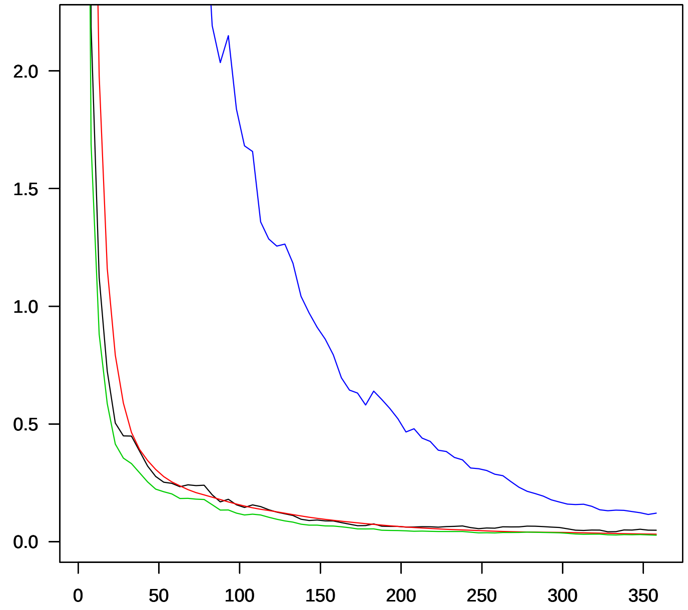




$\hat{\theta}_n^G$ (in black), $\hat{\theta}_n^Z$ (in red), $\hat{\theta}_n^{ML}$ (in green) and $\hat{\theta}_n^R$ (in blue)

Weibull $\mathcal{W}(0.25, 0.25) \longrightarrow \theta = 4,\ b(x) = 0$

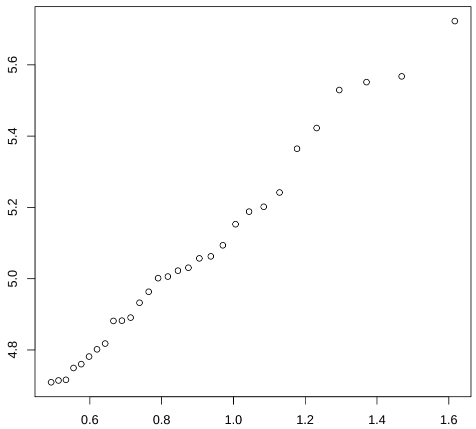




$\hat{\theta}_n^G$ (in black), $\hat{\theta}_n^Z$ (in red), $\hat{\theta}_n^{ML}$ (in green) and $\hat{\theta}_n^R$ (in blue)

6. Nidd river data.

- 154 exceedances of the level 65 $\mathrm{m}^3\mathrm{s}^{-1}$ by the river Nidd (Yorkshire, England) during the period 1934-1969 (35 years).
- Widely used in extreme value studies *i.e.* Hosking, Wallis, Wood (1985) and Davison, Smith (1990).
- The N-year return level is the water level which is exceeded on average once in N years.



QQ-plot

$\hat{k}_n = 29$,
$\hat{\theta}_n^{ML} \simeq 0.89$,

Estimation of the 100-year return level:
$\hat{x}_{p_n}(\hat{\theta}_n^{ML}) = 366 m^3 s^{-1}$

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